Stationary distributions of sums of marginally chaotic variables as renormalization group fixed points

Miguel A. Fuentes[1,2] **and A. Robledo**[3]

[1] Santa Fe Institute, 1399 Hyde Park Road, Santa Fe, New Mexico 87501, USA
[2] Centro Atómico Bariloche, Instituto Balseiro and CONICET, 8400 Bariloche, Argentina
[3] Instituto de Física, Universidad Nacional Autónoma de México, Apartado Postal 20-364, México 01000 DF, Mexico

E-mail: `fuentesm@santafe.edu`

Abstract. We determine the limit distributions of sums of deterministic chaotic variables in unimodal maps assisted by a novel renormalization group (RG) framework associated to the operation of increment of summands and rescaling. In this framework the difference in control parameter from its value at the transition to chaos is the only relevant variable, the trivial fixed point is the Gaussian distribution and a nontrivial fixed point is a multifractal distribution with features similar to those of the Feigenbaum attractor. The crossover between the two fixed points is discussed and the flow toward the trivial fixed point is seen to consist of a sequence of chaotic band mergers.

1. Introduction

The hegemony of the Central Limit Theorem [1, 2] for sums of deterministic variables generated by a number chaotic mappings have for some time been observed and also mathematically proved [3]. Since the mixing properties of chaotic trajectories yield variables indistinguishable to independent random variables, it is of interest to study nonmixing systems such as mappings at the transition from regular to chaotic behavior. Recent [4]-[7] numerical explorations of time averages of iterates at the period-doubling transition to chaos [8] have been presented and interpreted as possible evidence for a novel type of stationary distribution.

The dynamics toward and at the Feigenbaum attractor is now known in much detail [9, 10], therefore, it appears feasible to analyze also the properties of sums of iterate positions for this classic nonlinear system with the same kind of analytic reasoning and numerical thoroughness. Here we present the results for sums of chronological positions of trajectories associated to quadratic unimodal maps. We consider the case of the sum of positions of trajectories inside the Feigenbaum attractor as well as those within the chaotic 2^n-band attractors obtained when the control parameter is shifted to values larger than that at the transition to chaos. From the information obtained we draw conclusions on the properties of the stationary distributions for these sums of variables. Our results, that reveal a multifractal stationary distribution that mirrors the features of the Feigenbaum attractor, can be easily extended to other critical attractor universality classes and other routes to chaos.

The overall picture we obtain is effectively described within the framework of the renormalization group (RG) approach for systems with scale invariant states or attractors.

Firstly, the RG transformation for the distribution of a sum of variables is naturally given by the change due to the increment of summands followed by a suitable restoring operation. Second, the limit distributions can be identified as fixed points reached according to whether the acting relevant variables are set to zero or not. Lastly, the universality class of the non-trivial fixed-point distribution can be assessed in terms of the existing set of irrelevant variables.

As it is well known [8] a few decades ago the RG approach was successfully applied to the period-doubling route to chaos displayed by unimodal maps. In that case the RG transformation is functional composition of the mapping and its effect re-enacts the growth of the period doubling cascade. In our case the RG transformation is the growth and adjustment of the sum of positions and its effect is instead to go over again the merging of bands in the chaotic region.

Specifically, we consider the Feigenbaum map $g(x)$, obtained from the fixed point equation $g(x) = \alpha g(g(x/\alpha))$ with $g(0) = 1$ and $g'(0) = 0$, and where $\alpha = -2.50290...$ is one of Feigenbaum's universal constants [8]. For expediency we shall from now on denote the absolute value $|\alpha|$ by α. Numerically, the properties of $g(x)$ can be conveniently obtained from the logistic map $f_{\mu,2}(x) = 1 - \mu x^2$, $-1 \leq x \leq 1$, with $\mu = \mu_\infty = 1.401155189092...$ The dynamics associated to the Feigenbaum map is determined by its multifractal attractor. For a recent detailed description of these properties see [9, 10]. For values of $\mu > \mu_\infty$ we employ a well-known scaling relation supported by numerical results.

Initially we present properties of the sum of the absolute values $|x_t|$ of positions $x_t = f_{\mu_\infty,2}(x_{t-1})$, $t = 1, 2, 3, ...$, as a function of total time N visited by the trajectory with initial position $x_0 = 0$, and obtain a patterned linear growth with N. We analyze this intricate fluctuating pattern, confined within a band of finite width, by eliminating the overall linear increment and find that the resulting stationary arrangement exhibits features inherited from the multifractal structure of the attractor. We derive an analytical expression for the sum that corroborates the numerical results and provide an understanding of its properties. Next, we consider the straight sum of x_t, where the signs taken by positions lessen the growth of its value as N increases and the results are consistently similar to those for the sum of $|x_t|$, i.e. linear growth of a fixed-width band within which the sum displays a fluctuating arrangement. Further details for the sum of x_t are not included because of repetitiveness. Then, we show numerical results for the sum of iterated positions obtained when the control parameter is shifted into the region of chaotic bands. In all of these cases the distributions evolve after a characteristic crossover towards a Gaussian form. Finally, we rationalize our results in terms of an RG framework in which the action of the Central Limit Theorem plays a fundamental role.

The starting point of our study is evaluation of

$$y_\mu(N) \equiv \sum_{t=1}^{N} |x_t|, \tag{1}$$

with $\mu = \mu_\infty$ and with $x_0 = 0$. Fig. 1A shows the result, where it can be observed that the values recorded, besides a repeating fluctuating pattern within a narrow band, increase linearly on the whole. The measured slope of the linear growth is $c = 0.56245...$ Fig. 1B shows an enlargement of the band, where some detail of the complex pattern of values of $y_{\mu_\infty}(N)$ is observed. A stationary view of the mentioned pattern is shown in Fig. 1C, where we plot

$$y'_{\mu_\infty}(N) \equiv \sum_{t=1}^{N} (|x_t| - c), \tag{2}$$

in logarithmic scales. There, we observe that the values of $y'_{\mu_\infty}(N)$ fall within horizontal bands interspersed by gaps, revealing a fractal or multifractal set layout. The top (zeroth) band
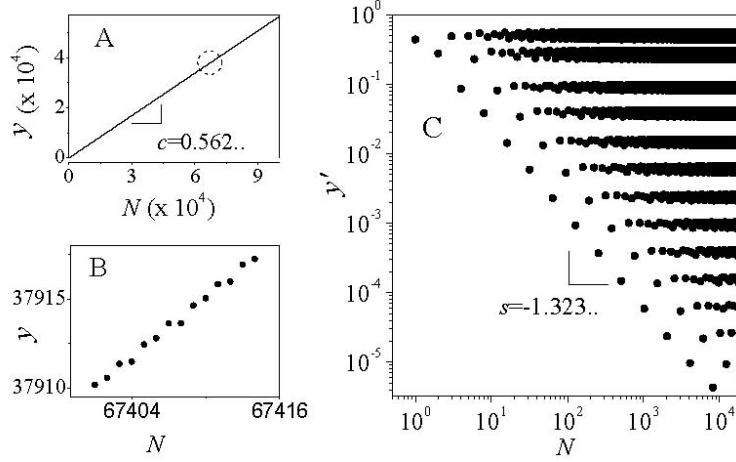
Figure 1. A) Sum of absolute values of visited points x_t, $t = 0, ..., N$, of the Feigenbaum's attractor with initial condition $x_0 = 0$. B) A closer look of the path of the sum (see dotted circle in A), for values of N around 67410. C) Centered sum $y'(N)$ in logarithmic scales. See text.

contains y'_{μ_∞} for all the odd values of N, the 1st band next to the top band contains y'_{μ_∞} for the even values of N of the form $N = 2 + 4m$, $m = 0, 1, 2, ...$ The 2nd band next to the top band contains $y'_{\mu_\infty}(N)$ for $N = 2^2 + 2^3 m$, $m = 0, 1, 2, ...$, and so on. In general, the k-th band next to the top band contains $y'_{\mu_\infty}(2^k + 2^{k+1}m)$, $m = 0, 1, 2, ...$ Another important feature in this figure is that the $y'_{\mu_\infty}(N)$ for subsequences of N each of the form $N = (2l + 1)2^k$, $k = 0, 1, 2, ...$, with l fixed at a given value of $l = 0, 1, 2, ...$, appear aligned with a uniform slope $s = -1.323...$ The parallel lines formed by these subsequences imply the power law $y'_{\mu_\infty}(N) \sim N^s$ for N belonging to such a subsequence.

It is known [9, 11] that these two characteristics of $y'_{\mu_\infty}(N)$ are also present in the layout of the absolute value of the individual positions $|x_t|$, $t = 1, 2, 3, ...$ of the trajectory initiated at $x_0 = 0$; and this layout corresponds to the multifractal geometric configuration of the points of the Feigenbaum's attractor, see Fig. 1 in [11]. In this case, the horizontal bands of positions separated by equally-sized gaps are related to the period-doubling 'diameters' [8] set construction of the multifractal [10]. The identical slope shown in the logarithmic scales by all the position subsequences $|x_t|$, $t = (2l + 1)2^k$, $k = 0, 1, 2, ...$, each formed by a fixed value of $l = 0, 1, 2, ...$, implies the power law $|x_t| \sim t^s$, $s = -\ln \alpha / \ln 2 = -1.3236...$, as the $|x_t|$ can be expressed as $|x_t| \simeq |x_{2l+1}| \ \alpha^{-k}, t = (2l + 1)2^k$, $k = 0, 1, 2, ...$, or, equivalently, $|x_t| \sim t^s$. Notice that the index k also labels the order of the bands from top to bottom. The power law behavior involving the universal constant α of the subsequence positions reflect the approach of points in the attractor toward its most sparse region at $x = 0$ from its most compact region, as the positions at odd times $|x_{2l+1}| = x_{2l+1}$, those in the top band, correspond to the densest region of the set.

Having uncovered the through manifestation of the multifractal structure of the attractor into the sum $y'_{\mu_\infty}(N)$ we proceed to derive this property and corroborate the numerical evidence. Consider Eq. (1) with $N = 2^k$, $k = 0, 1, 2, ...$, the special case $l = 0$ in the discussion above. Then the numbers of terms $|x_t|$ per band in $y_{\mu_\infty}(2^k)$ are: 2^{k-1} in the top band ($j = 0$), 2^{k-2} in the next band ($j = 1$),..., 2^0 in the $(k - 1)$-th band, plus an additional position in the k-th

band. If we introduce the average of the positions on the top band

$$\langle a \rangle \equiv 2^{-(k-1)} \sum_{j=0}^{2^{k-1}} x_{2j+1}, \tag{3}$$

the sum $y_{\mu_\infty}(2^k)$ can be written as

$$y_{\mu_\infty}(2^k) = \langle a \rangle \, 2^{k-1} \sum_{j=0}^{k-2} (2\alpha)^{-j} + \alpha^{-(k-1)} + \alpha^{-k}. \tag{4}$$

Doing the geometric sum above and expressing the result as $y_{\mu_\infty}(2^k) = c2^k + d\alpha^{-k}$, we have

$$c = \frac{\langle a \rangle \, \alpha}{2\alpha - 1}, \quad d = \left(1 - \frac{\langle a \rangle \, 2\alpha}{2\alpha - 1}\right) \alpha + 1 . \tag{5}$$

Evaluation of Eq. (3) yields to $\langle a \rangle = 0.8999...$, and from this we obtain $c = 0.56227...$ and $d = 0.68826...$ We therefore find that the value of the slope c in Fig. 1A is properly reproduced by our calculation. Also, since $\ln\left[y_{\mu_\infty}(2^k) - c2^k\right] = \ln d - k \ln \alpha$, or, equivalently, $\ln y'_{\mu_\infty}(N) = \ln d - N \ln \alpha / \ln 2$, $N = 2^k$, $k = 0, 1, 2, ...$, we corroborate that the value of the slope s in inset of Fig. 1C is indeed given by $s = -\ln \alpha / \ln 2 = 1.3236...$ (We have made use of the identity $\alpha^{-k} = N^{-\ln \alpha / \ln 2}$, $N = 2^k$, $k = 0, 1, 2, ...$).

We note that the sum of x_t from $t = 0$ to $N = 2^k$, i.e. considering the signs taken by positions, can be immediately obtained from the above by replacing α^{-j} by $(-1)^j \alpha^{-j}$ as the x_t of different signs of the trajectory starting at $x_0 = 0$ fall into separate alternating bands (described above and shown in Fig. 1 of [11]). In short, $x_t \simeq (-1)^j x_{2l+1} \, \alpha^{-j}$, $t = (2l+1)2^k$, $k = 0, 1, 2, ...$ As stated, our numerical and analytical results are in agreement also in this case.

We turn now to study the sum of positions of trajectories when $\Delta\mu \equiv \mu - \mu_\infty > 0$. We recall that in this case the attractors are made up of 2^K, $K = 1, 2, 3, ...$, bands and that their trajectories consist of an interband periodic motion of period 2^K and an intraband chaotic motion. We evaluated numerically the sums $y_\mu(N)$ for an ensemble of initial conditions x_0 uniformly distributed only within the chaotic bands, for different values of $\Delta\mu$; $y'_\mu(N)$ was then obtained similarly to Eq. (2) by substracting the average $\langle y_\mu(N) \rangle_{x_0}$ and rescaling with a factor $N^{-1/2}$. The panels in Fig. 2 show the evolution of the distributions for increasing number of summands N for a value of $\Delta\mu$ (chosen for visual clarity) when the attractor consists of 2^3 chaotic bands. Initially the distributions are multimodal with disconnected domains, but as N increases we observe merging of bands and development of a single-domain bell-shaped distribution that as $N \longrightarrow \infty$ converges in all cases to a Gaussian distribution.

These numerical results can be understood as follows. We recollect [8] that the relationship between the number 2^K, $K \gg 1$, of bands of a chaotic attractor and the control parameter distance $\Delta\mu$ at which it is located is given by $2^K \sim \Delta\mu^{-\kappa}$, $\kappa = \ln 2 / \ln \delta_F$, where $\delta_F = 0.46692...$ is the universal constant that measures both the rate of convergence of the values of μ at period doublings or at band splittings to μ_∞. For $\Delta\mu$ small and fixed, the sum of sequential positions of the trajectory initiated at $x_0 = 0$, Eq. (1), exhibits two different growth regimes as the total time N increases. In the first one, when $N \ll 2^K$, the difference in value $\delta x_t \equiv x_t(\mu) - x_t(\mu_\infty)$ between the positions at time t for μ and μ_∞ do not affect qualitatively the multifractal structure of the sum at μ_∞, nor its associated distribution. This is because the fine structure of the Feigenbaum attractor is not suppressed by the fluctuations δx_t, as these contribute to the sum individually during the first cycle of the interband periodic motion. The discrete multi-scale nature of the distribution for μ_∞ is preserved when the interband motion governs the sum
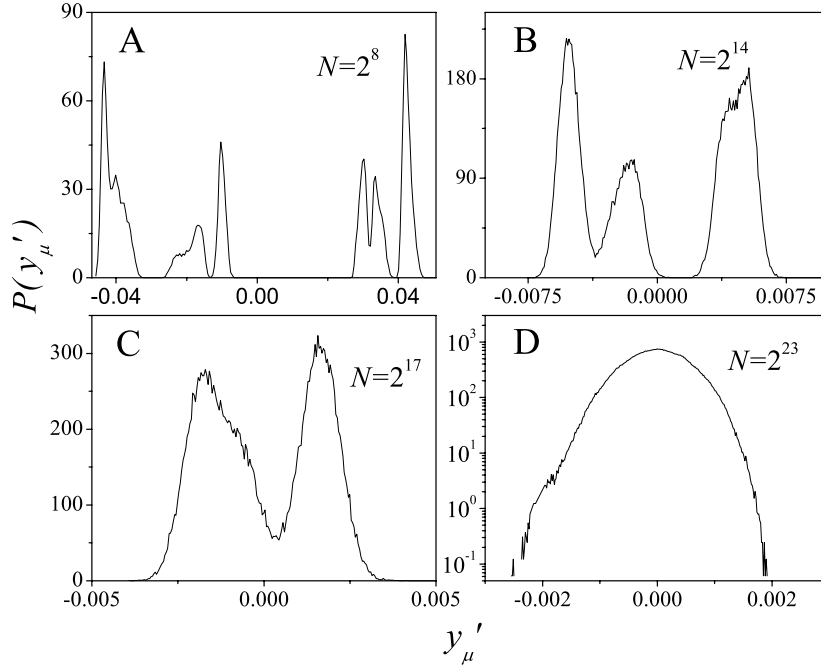
Figure 2. Distributions for the sums of $|x_t|$, $t = 0, ..., N$, of an ensemble of trajectories with initial conditions within the 2^3-band attractor at $\Delta\mu = 0.0028448109$. The number of summands N are indicated in each panel. See text.

$y_\mu(N)$. In the second regime, when $N \gg 2^K$, the situation is opposite, after many interband cycles the fluctuations δx_t add up in the sum and progressively wipe up the fine structure of the Feigenbaum attractor, leading to merging of bands and to the dominance of the fluctuating intraband motion. Ultimately, as $N \longrightarrow \infty$ the evolution of the distribution is similar to the action of the Central Limit Theorem and leads to a Gaussian stationary end result. It is also evident that as $\Delta\mu$ increases the first regime is shortened at the expense of the second, whereas when $\Delta\mu \longrightarrow 0$ the converse is the case. Therefore there exists an unambiguous $\Delta\mu$-dependent crossover behavior between the two radically different types of stationary distributions. This crossover is set out when the δx_t fluctuations begin removing the band structure in $y'_\mu(N)$ when $\Delta\mu$ is small and ends when these fluctuations have broadened and merged all the chaotic bands and $y'_\mu(N)$ forms a single continuous interval. When $\mu = \mu_\infty$ this process never takes place.

We are in a position now to put together the numerical and analytical information presented above into the general framework of the RG approach. As known, this method was designed to characterize families of systems containing amongst their many individual states (or in this case attractors) a few exceptional ones with scale invariant properties and common to all systems in the family. We recall [12] that in the language of a minimal RG scheme there are two fixed points, each of which can be reached by the repeated application of a suitable transformation of the system's main defining property. One of the fixed points, is termed trivial and is reached via the RG transformation for almost all initial settings. i.e. for all systems in the family when at least one of a small set of variables, named relevant variables, is nonzero. To reach the other fixed point, termed nontrivial, it is necessary that the relevant variables are all set to zero, and this implies a severely restricted set of initial settings that ensure such critical RG paths. The nontrivial fixed point embodies the scale invariant properties of the exceptional state that occurs in each system in the family and defines a universality class, while the differences amongst the

individual systems are distinguished through a large set of so-called irrelevant variables. The variables in the latter set gradually vanish as the RG transformation is applied to a system that evolves toward the nontrivial fixed point. Further, when any system in the family is given a nonzero but sufficiently small value to (one or more of) the relevant variables, the RG transformation converts behavior similar to that of the nontrivial fixed point into that resembling the trivial fixed point through a well-defined crossover phenomenon. The recognition of the RG framework in the properties of the sums of positions of trajectories in unimodal maps and their associated distributions is straightforward. It can be concluded right away that in this problem (as defined here) there is only one relevant variable, the control parameter difference $\Delta\mu$. There is an infinite number of irrelevant variables, those that specify the differences between all possible unimodal maps (with quadratic maximum) and the Feigenbaum map $g(x)$. There are two fixed-point distributions, the trivial continuum-space Gaussian distribution and the nontrivial discrete-space multifractal distribution (as observed in Fig. 1C). As explained above, there is a distinct crossover link between the two fixed-point distributions. The RG transformation consists of the increment of one or more summands in the sum (1) followed by centering like in Eq. (2). The effect of the transformation in the distribution of the sum is then recorded. For sums of independent variables the transformation is equivalent to the convolution of distributions. Our results correspond to the dynamics inside the atractors, however, if the interest lies in considering only the stationary distribution of sums that do not contain the transient behavior of trajectories in their way to the attractor [4] our results are expected to give the correct answers for this case.

In summary, we have found that the stationary distribution of the sum of iterate positions within the Feigenbaum attractor has a multifractal structure stamped by that of the initial multifractal set, while that involving sums of positions within the attractors composed of 2^K chaotic bands is the Gaussian distribution. We have also shown that the entire problem can be couched in the language of the RG formalism in a way that makes clear the identification of the existing stationary distributions and the manner in which they are reached. These basic features suggest a degree of universality, limited to the critical attractor under consideration, in the properties of sums of deterministic variables at the transitions to chaos. Namely, the sums of positions of memory-retaining trajectories evolving under a vanishing Lyapunov exponent appear to preserve the particular features of the multifractal critical attractor under examination. Thus we expect that varying the degree of nonlinearity of a unimodal map would affect the scaling properties of time averages of trajectory positions at the period doubling transition to chaos, or alternatively, that the consideration of a different route to chaos, such as the quasiperiodic route, would lead to different scaling properties of comparable time averages. For instance, the known dependence of the universal constant α on the degree of nonlinearity z of a unimodal map would show as a z-dependent exponent $s = \ln\alpha/\ln 2$ that controls the scale invariant property of the sum of trajectory positions with $x_0 = 0$ (shown in Fig. 1C).

We analyzed the nature and the conditions under which a stationary distribution with universal properties (in the Renormalization Group sense) occurs for sums of deterministic variables at the transition between regular and chaotic behavior, such as those studied here for dynamics at zero Lyapunov exponent. The nonexistence of fluctuations in such critical attractors implies a distribution of the sums of these variables strictly defined on a discrete multifractal set and therefore different from known (Gaussian or otherwise) continuum-space limit distributions for real number random variables.

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